A New Acceleration Factor Decision Method for ICCG Method Utilizing Quality of Finite Element Mesh

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Abstract — ICCG method is widely used to solve a sparse symmetric linear system that results from application of finite element analysis. In order to improve the convergence property of the ICCG method, an acceleration factor was proposed. Then its automatic decision method has been already proposed. However, by using the method, much more iterations are sometimes required, compared with using the optimal acceleration factor. In this paper, we have investigated on the optimal acceleration factor to develop a more accurate acceleration factor decision method.

I. INTRODUCTION

Finite element method (FEM) yields a sparse symmetric linear system. In the finite element analysis, the ICCG method is widely used to solve it. In order to solve it with short time, an acceleration factor of the ICCG method was proposed [1]. The effectiveness and usefulness of the acceleration factor was demonstrated in [1]-[2]. However, it is known that a residual may not be converged or a needlessly large number of iterations of the ICCG method are required unless the acceleration factor is optimally determined. Therefore, a way to find the optimal acceleration factor, which minimizes the number of iterations, would be expected. Conventionally the paper [2] proposed an acceleration factor decision method by finding the minimum of the maximum diagonal entries of the lower triangular matrix.

Then, in order to investigate the usefulness of the conventional acceleration factor decision method [2], we have employed a simple test problem and a practical model. From these results, a new method to find the optimal acceleration factor was required. In this paper, we propose the results of the investigations in order to develop a new acceleration factor decision method. We have calculated the condition number of the pre-conditioned system after the incomplete Cholesky decomposition and investigated a correlation of the condition number and the optimal acceleration factor.

II. ACCELERATION FACTOR OF ICCG METHOD

The acceleration factor is introduced into the incomplete Cholesky decomposition (IC decomposition), as follows:

$$L_{\mathrm{ij}} = \begin{cases} \gamma H_{\mathrm{ii}} - \sum_{\mathrm{k}=1}^{\mathrm{i}-1} L_{\mathrm{ik}}{}^{2} D_{\mathrm{kk}} & (\mathrm{i}=\mathrm{j}) \\ H_{ii} - \sum_{\mathrm{k}=1}^{\mathrm{i}-1} L_{\mathrm{ik}} L_{\mathrm{jk}} D_{\mathrm{kk}} & (\mathrm{i}>\mathrm{j}) \end{cases}, \tag{1}$$

$$D_{\mathrm{ii}} = 1/L_{\mathrm{ii}} \ , \tag{2}$$

where L_{ij} and D_{ii} are the entries in $\boldsymbol{L}$ and $\boldsymbol{D}$, respectively. When γ is equal to 1.0, the above decomposition corresponds to the normal IC decomposition.

III. PRE-CONDITIONED SYSTEM

Given the linear system

$$\boldsymbol{Ax} = \boldsymbol{b} \ , \tag{3}$$

where $\boldsymbol{A}$ is the coefficient matrix, $\boldsymbol{x}$ is the solution vector and $\boldsymbol{b}$ is the right hand vector. Using the IC decomposition of $\boldsymbol{A}$, $\boldsymbol{A} = \boldsymbol{LDL}^T$, the system (3) is pre-conditioned as follows:

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{x}} = \tilde{\boldsymbol{b}} \ , \tag{4}$$

where

$$\tilde{\boldsymbol{A}} = \boldsymbol{M}^{-1}\boldsymbol{A}\boldsymbol{M}^{-T}, \ \tilde{\boldsymbol{x}} = \boldsymbol{M}^{T}\boldsymbol{x}, \ \tilde{\boldsymbol{b}} = \boldsymbol{M}^{-1}\boldsymbol{b}, \tag{5}$$

$$\boldsymbol{M} = \boldsymbol{LD}^{1/2} \ . \tag{6}$$

The pre-conditioned system (4) depends on the acceleration factor γ since $\boldsymbol{L}$ and $\boldsymbol{D}$ is obtained using the acceleration factor. We calculated the condition number κ of $\tilde{\boldsymbol{A}}$ as follows [3] :

$$\kappa = \frac{\mu_{\max}}{\mu_{\min}} \ , \tag{7}$$

where $\mu_{\max}$ is the maximum eigenvalue of $\tilde{\boldsymbol{A}}$ and $\mu_{\min}$ is the nonzero minimum eigenvalue.

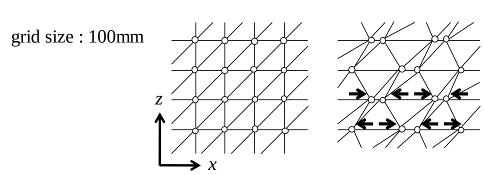
IV. INVESTIGATION OF CONVERGENCE PROPERTY

A. *Simple Test Model*

The simple test model consists of a permanent magnet (1.0 T). The analysis region is firstly divided into regular hexahedra, and then each hexahedron is divided into six tetrahedra. The properties of the test model are shown in Table I. For the investigation, distorted meshes are generated by zigzag moving the nodes of the test model into *x*-direction, as shown in Fig. 1. The moving distance of nodes is varied from 5.00 to 49.99 mm.

TABLE I
MESH PROPERTIES

Nodes	Elements	Edges of unknown
1,331	6,000	4,200



(a) (b) distorted mesh
Fig. 1. Conceptual illustration of meshes.

Fig. 2 shows the optimal acceleration factors γ_{opt} and the automatically decided ones γ_{auto} versus the moving distance *d* and Fig. 3 shows the number of ICCG iterations with both the acceleration factors γ_{opt} and γ_{auto}.

Fig. 2 shows that γ_{opt} is equal to or larger than γ_{auto}. The larger γ_{opt} is, the larger the distance *d* is. Then, Fig. 3 shows that when *d* is larger than 49.00 mm so that the quality of the mesh is much worse, the difference of the ICCG iterations for γ_{opt} and γ_{auto} expands. On the other hand, when the mesh is well shaped (*d* = 5.00 and 10.00 mm), the acceleration factors γ_{opt} and γ_{opt} are almost the same.

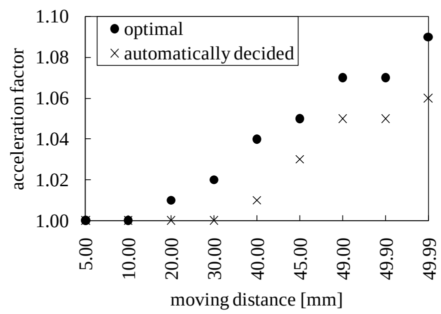


Fig. 2. Acceleration factors.

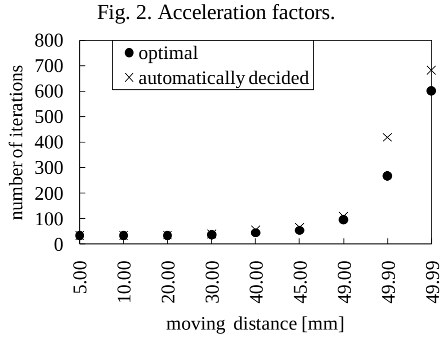


Fig. 3. Number of ICCG iterations.

B. SQUID Model

Next, as a practical and complicated model, a SQUID model (thin plate model) [4], as shown in Fig. 4, is solved by FEM. The mesh properties of SQUID model are shown in Table II, and the mesh includes many ill-shaped elements, such as a flat element. Fig. 5 shows the number of ICCG iterations and the maximum diagonal entry of the lower triangular matrix $\boldsymbol{L}$, which is used in order to estimate γ_{auto} by the automatic decision method, versus the acceleration factor γ.

Fig. 5 shows that γ_{opt} is much larger than γ_{auto}. It is also larger than that of the simple test model. Then, the large γ_{opt} of the SQUID model is due to the poor quality of the finite element mesh because the mesh of the SQUID model includes much more thin-shaped elements than the simple test model. From these results, the optimal acceleration factor γ_{opt} depends on the quality of the finite element mesh. Therefore, the method to estimate γ_{opt} by evaluating the quality of the finite element mesh will be proposed in the full paper.

Table II.
MESH PROPERTIES OF SQUID MODEL

Nodes	Elements	Edges of unknown
337,837	2,303,465	1,948,973

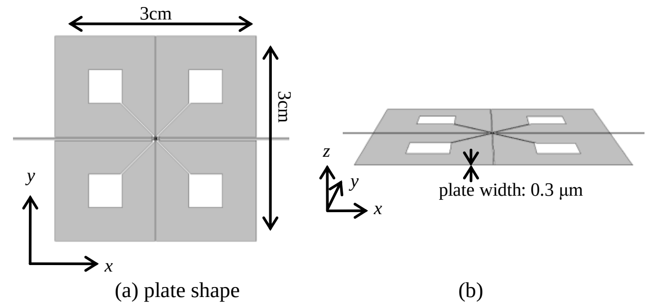


Fig. 4. SQUID model [4].

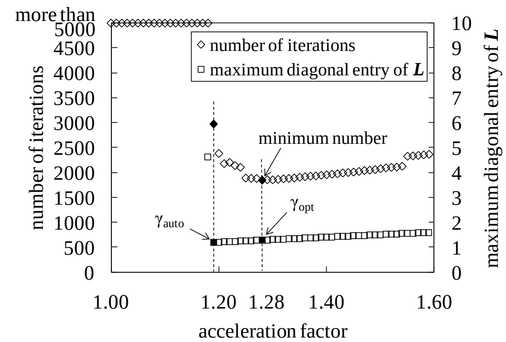


Fig. 5. Results of SQUID model [4].

V. CONDITION NUMBER OF PRE-CONDITIONED SYSTEM

Fig. 6 shows the condition number κ of $\tilde{\boldsymbol{A}}$ and the number of ICCG iterations versus the acceleration factor for the simple test model.

Notice that the acceleration factor, that yields the minimum of the condition number κ, corresponds with the optimal acceleration factor. Consequently, the optimal acceleration factor can be decided by finding the minimum condition value.

It is necessary to develop the easy way to calculate the condition number for the large-scale matrix.

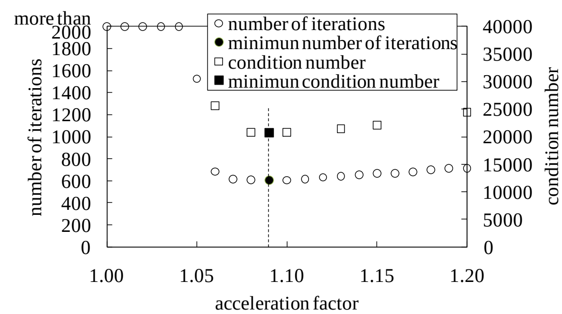


Fig. 6. Condition number of pre-conditioned system.

VI. REFERENCES

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